ON LINEAR RELATIONS FOR DIRICHLET SERIES FORMED BY RECURSIVE SEQUENCES OF SECOND ORDER

Carsten Elsner and Niclas Technau

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Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Four corresponding zeta functions in s are defined by

$$\zeta_F(s) := \sum_{n=1}^{\infty} \frac{1}{F_n^s}, \quad \zeta_F^*(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n^s}, \quad \zeta_L(s) := \sum_{n=1}^{\infty} \frac{1}{L_n^s}, \quad \zeta_L^*(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_n^s}.$$

As a consequence of Nesterenko's proof of the algebraic independence of the three Ramanujan functions $R(\rho), Q(\rho), P(\rho)$ for any algebraic number ρ with $0 < \rho < 1$, the algebraic independence or dependence of various sets of these numbers are already known for positive even integers s. In this paper, we investigate linear forms in the above zeta functions and determine the dimension of linear spaces spanned by such linear forms. In particular, it is established that for any positive integer m the solutions of

$$\sum_{s=1}^{m} \big(t_s\zeta_F(2s) + u_s\zeta_F^*(2s) + v_s\zeta_L(2s) + w_s\zeta_L^*(2s) \big) = 0$$

with $t_s, u_s, v_s, w_s \in \mathbb{Q}$ $(1 \leq s \leq m)$ form a $\mathbb{Q}$-vector space of dimension m. This proves a conjecture from the Ph.D. thesis of M. Stein, who, in 2012, was inspired by the relation $-2\zeta_F(2) + \zeta_F^*(2) + 5\zeta_L^*(2) = 0$. All the results are also true for zeta functions in $2s$ where the Fibonacci and Lucas numbers are replaced by numbers from sequences satisfying a second order recurrence formula. Again, our results are based on the algebraic independence of the Ramanujan functions.

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