ALGEBRAIC RESULTS FOR CERTAIN VALUES OF THE JACOBI THETA-CONSTANT $\theta_3(\tau)$

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In its most elaborate form, the Jacobi theta function is defined for two complex variables z and τ by $\theta(z|\tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i\nu^2\tau+2\pi i\nu z}$, which converges for all complex number z, and τ in the upper half-plane. The special case

$$\theta_3(\tau) := \theta(0|\tau) = 1 + 2\sum_{\nu=1}^{\infty} e^{\pi i\nu^2\tau}$$

is called a Jacobi theta-constant or Thetanullwert of the Jacobi theta function $\theta(z|\tau)$. In this paper, we prove the algebraic independence results for the values of the Jacobi theta-constant $\theta_3(\tau)$. For example, the three values $\theta_3(\tau)$, $\theta_3(n\tau)$, and $D\theta_3(\tau)$ are algebraically independent over $\mathbb{Q}$ for any τ such that $q = e^{i\pi\tau}$ is an algebraic number, where $n \geq 2$ is an integer and $D := (\pi i)^{-1}d/d\tau$ is a differential operator. This generalizes a result of the first author, who proved the algebraic independence of the two values $\theta_3(\tau)$ and $\theta_3(2^m\tau)$ for $m \geq 1$. As an application of our main theorem, the algebraic dependence over $\mathbb{Q}$ of the three values $\theta_3(\ell\tau)$, $\theta_3(m\tau)$, and $\theta_3(n\tau)$ for integers $\ell, m, n \geq 1$ is also presented.

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