ALGEBRAIC INDEPENDENCE RESULTS FOR VALUES OF THETA-CONSTANTS, II

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Let $\theta_3(\tau) = 1 + 2\sum_{\nu=1}^{\infty} q^{\nu^2}$ with $q = e^{i\pi\tau}$ denote the Thetanullwert of the Jacobi theta function

$$\theta(z|\tau) = \sum_{\nu=-\infty}^{\infty} e^{\pi i\nu^2\tau + 2\pi i\nu z} .$$

Moreover, let $\theta_2(\tau) = 2\sum_{\nu=0}^{\infty} q^{(\nu+1/2)^2}$ and $\theta_4(\tau) = 1 + 2\sum_{\nu=1}^{\infty} (-1)^\nu q^{\nu^2}$. For algebraic numbers q with $0 < |q| < 1$ and for any $j \in \{2,3,4\}$ we prove the algebraic independence over $\mathbb{Q}$ of the numbers $\theta_j(n\tau)$ and $\theta_j(\tau)$ for all odd integers $n \geq 3$. Assuming the same conditions on q and τ as above, we obtain sufficient conditions by use of a criterion involving resultants in order to decide on the algebraic independence over $\mathbb{Q}$ of $\theta_j(2m\tau)$ and $\theta_j(\tau)$ $(j = 2,3,4)$ and of $\theta_3(4m\tau)$ and $\theta_3(\tau)$ with odd positive integers m. In particular, we prove the algebraic independence of $\theta_3(n\tau)$ and $\theta_3(\tau)$ for even integers n with $2 \leq n \leq 22$. The paper continues the work of the first-mentioned author, who already proved the algebraic independence of $\theta_3(2^m\tau)$ and $\theta_3(\tau)$ for $m = 1, 2, \ldots$.

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