ON ERROR SUMS FOR SQUARE ROOTS OF POSITIVE INTEGERS WITH APPLICATIONS TO LUCAS AND PELL NUMBERS

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Let p_m/q_m denote the convergents of a real number α. An error sum is a series of the form $\sum_{m=0}^{\infty} f(\alpha q_m - p_m)$, where f denotes a real function defined on the interval $[0,1]$. In this paper we mainly consider the error sums

$$\mathcal{E}(\sqrt{D}) = \sum_{m=0}^{\infty} |q_m\sqrt{D} - p_m| \quad \text{and} \quad \mathcal{E}^*(\sqrt{D}) = \sum_{m=0}^{\infty} (q_m\sqrt{D} - p_m)\,,$$

where $D \neq 1^1, 2^2, 3^2, \ldots$ is a positive integer. We express these sums in terms of a finite number of convergents of $\sqrt{D}$. These identities imply among other things that $\mathcal{E}(\sqrt{D}) + \mathcal{E}^*(\sqrt{D}) \in \mathbb{Q}$. In particular, for $D = 5$ and $D = 2$ we obtain the formulas

$$\frac{5}{2} = \sum_{m=0}^{\infty} \left(\left(L_{6m+2} + L_{6m+4}\right)\sqrt{5} - 5L_{6m+3} \right)\,, \qquad \frac{1}{2} = \sum_{m=0}^{\infty} \left(P_{2m+1}\sqrt{2} - P_{2m} - P_{2m+1} \right)\,,$$

where L_n and P_n are the Lucas and Pell numbers, respectively.

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