ON RECIPROCAL SUMS FORMED BY SOLUTIONS OF PELL'S EQUATION

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Let $(X_n, Y_n)_{n\geq 1}$ denote the positive integer solutions of Pell's equation $X^2 - DY^2 = 1$ or $X^2 - DY^2 = -1$. We introduce the Dirichlet series $\zeta_X(s) = \sum_{n=1}^{\infty} 1/X_n^s$ and $\zeta_Y(s) = \sum_{n=1}^{\infty} 1/Y_n^s$ for $\Re(s) > 0$ and prove that both functions do not satisfy any nontrivial algebraic differential equation. For any positive integers s_1 and s_2 the two numbers $\zeta_X(2s_1)$ and $\zeta_Y(2s_2)$ are algebraically independent over a transcendental field extension of $\mathbb{Q}$, whereas the three numbers $\zeta_X(2)$, $\zeta_Y(2)$, and $\sum_{n=1}^{\infty} 1/(X_nY_n)^2$ are algebraically independent over $\mathbb{Q}$. From the transcendence of $\zeta_Y(2)$ and the corresponding alternating series we obtain an application to the Archimedean cattle problem. Irrationality results for series of the form $\sum_{n=1}^{\infty} (-1)^{n+1}/X_n$, $\sum_{n=1}^{\infty} (-1)^{n+1}/Y_n$, and $\sum_{n=1}^{\infty} (-1)^{n+1}/X_nY_n$ are obtained by a theorem of R. André-Jeannin.

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