ALGEBRAIC INDEPENDENCE RESULTS FOR VALUES OF THETA-CONSTANTS

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Functiones et Approximatio 52.1 (2015), 7-27.
DOI: 10.7169/facm/2015.52.1.1

Let $\theta(q) = 1 + 2\sum_{\nu=1}^{\infty} q^{\nu^2}$ denote the Thetanullwert of the Jacobi Zeta function

$$\theta(z|\tau) \,=\, \sum_{\nu=-\infty}^{\infty} e^{\pi i\nu^2\tau+2\pi i\nu z}\,.$$

For algebraic numbers q with $0 < |q| < 1$ we prove the algebraic independence over $\mathbb{Q}$ of the numbers $\theta(q^n)$ and $\theta(q)$ for $n = 2, 3, \ldots, 12$ and furthermore for all $n \geq 16$ which are powers of two. An application for $n = 5$ proves the transcendence of the number

$$\sum_{j=1}^{\infty} (-1)^j \left(\frac{j}{5}\right) \frac{jq^j}{1-q^j}\,.$$

Similar results are obtained for numbers related to modular equations of degree 3, 5, and 7.

MR 2010 Subject Classification: 11J85, 11J91, 11F27

Key words: Algebraic independence, Theta-constants, Nesterenko's theorem, Independence criterion, Modular equations