ON ERROR SUM FUNCTIONS FOR APPROXIMATIONS WITH ARITHMETIC CONDITIONS

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Let $\mathcal{E}_{k,l}(\alpha) = \sum_{q_m \equiv l \pmod k} |q_m\alpha - p_m|$ be error sum functions formed by convergents p_m/q_m ($m \geq 0$) of a real number α satisfying the arithmetical condition $q_m \equiv l \pmod k$ with $0 \leq l < k$. The functions $\mathcal{E}_{k,l}$ are Riemann-integrable on $[0,1]$, so that the integrals $\int_0^1 \mathcal{E}_{k,l}(\alpha)\, d\alpha$ exist as the arithmetical means of the functions $\mathcal{E}_{k,l}$ on $[0,1]$. We express these integrals by multiple sums on rational terms and prove upper and lower bounds. In the case when l vanishes (i.e., k divides q_m) and when the smallest prime divisor p_1 of $k = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ satisfies $p_1 > k^{\varepsilon}$ for some positive real number ε, we have found an asymptotic expansion in terms of k, namely $\int_0^1 \mathcal{E}_{k,0}(\alpha)\, d\alpha = \zeta(2)\big(2\zeta(3)k^2\big)^{-1} + \mathcal{O}\big(3^t k^{-2-\varepsilon}\big)$. This result includes all integers k which are of the form $k = p^a$ for primes p and integers $a \geq 1$.

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