ALGEBRAIC INDEPENDENCE OF CERTAIN NUMBERS RELATED TO MODULAR FUNCTIONS

Carsten Elsner, Shun Shimomura, and Iekata Shiokawa

Functiones et Approximatio, 47.1 (2012), 121–141
DOI: 10.7169/facm/2012.47.1.10

In previous papers the authors established a method how to decide on the algebraic independence of a set $\{y_1, \ldots, y_n\}$ when these numbers are connected with a set $\{x_1, \ldots, x_n\}$ of algebraic independent parameters by a system $f_i(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0$ $(i = 1, 2, \ldots, n)$ of rational functions. Constructing algebraic independent parameters by Nesterenko's theorem, the authors successfully applied their method to reciprocal sums of Fibonacci numbers and determined all the algebraic relations between three q-series belonging to one of the sixteen families of q-series introduced by Ramanujan.
In this paper we first give a short proof of Nesterenko's theorem on the algebraic independence of π, $e^{\pi\sqrt{d}}$ and a product of Gamma-values $\Gamma(m/n)$ at rational points m/n. Then we apply the method mentioned above to various sets of numbers. Our algebraic independence results include among others the coefficients of the series expansion of the Heumann-Lambda function, the values $P(q^r), Q(q^r)$, and $R(q^r)$ of the Ramanujan functions P, Q, and R, for $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$ and $r = 1, 2, 3, 5, 7, 10$, and the values given by reciprocal sums of polynomials.

MR 2000 Subject Classification: Primary: 11J85, Secondary: 11J89, 11J91, 11F03.

Key words: Algebraic independence, Ramanujan functions, Nesterenko's theorem, Complete elliptic integrals, Gamma function