

ON THE RESIDUE CLASSES OF INTEGER SEQUENCES SATISFYING A LINEAR THREE TERM RECURRENCE FORMULA

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In this paper we investigate linear three-term recurrence formulae $Z_n = T(n)Z_{n-1} + U(n)Z_{n-2}$ ($n \geq 2$) with sequences of integers $(T(n))_{n \geq 0}$ and $(U(n))_{n \geq 0}$, which are ultimately periodic modulo m , e.g.

$$\begin{aligned}(T(n) \bmod m)_{n \geq 0} &= (a_0, a_1, a_2, \dots, a_\rho, \overline{T_1, T_2, \dots, T_w}) , \\ (U(n) \bmod m)_{n \geq 0} &= (b_0, b_1, b_2, \dots, b_\rho, \overline{U_1, U_2, \dots, U_w}) .\end{aligned}$$

In a former paper the authors computed explicitly the coefficients of a linear three-term recurrence formula for $z_n = Z_{rn+i}$ with $0 \leq i < r$, when $(T(n))_{n \geq 0}$ and $(U(n))_{n \geq 0}$ belong to regular or non-regular Hurwitz-type continued fraction expansions. Using this result we show now that the sequence $(Z_n)_{n \geq 0}$ is ultimately periodic modulo m . As a consequence, for Hurwitz-type continued fraction expansions $\alpha = [a_0; \overline{T_1(k), \dots, T_r(k)}]_{k=1}^\infty$ or $\alpha = [a_0; a_1, \overline{T_1(k), \dots, T_r(k)}]_{k=1}^\infty$ with polynomials $T_1 \neq \text{const.}, T_2, \dots, T_r$ we deduce for all positive integers a and m that $\liminf_q \|q\alpha\| = 0$, where $q \equiv a \pmod m$ and $\|\cdot\|$ denotes the distance from the nearest integer. Finally, we are particularly interested in the recurrence formula $Z_n = (an + b)Z_{n-1} + cZ_{n-2}$ ($n \geq 2$) and compute the length of a period of the sequence $(Z_n \bmod m)_{n \geq 0}$, when (a, m) divides b , and $(c, m) = 1$. This generalizes former results of the authors dealing with regular continued fraction expansions of the numbers $\exp(1/s)$ for integers $s \geq 1$.

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