EXCEPTIONAL ALGEBRAIC RELATIONS FOR RECIPROCAL SUMS OF FIBONACCI AND LUCAS NUMBERS

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Let $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ be the sequences of Fibonacci and Lucas numbers, respectively. Moreover, we define a set Γ of 12 real numbers by

$$\Gamma := \left\{ \sum_{n=1}^{\infty} \frac{1}{F_n^{2s}} \,,\, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n^{2s}} \,,\, \sum_{n=1}^{\infty} \frac{1}{L_n^{2s}} \,,\, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_n^{2s}} \quad (s=1,2,3) \right\} .$$

From former investigations of the authors it is known that every 4-subset of Γ consisting of four numbers is algebraically dependent over $\mathbb{Q}$. It is also proven by the authors that each of the 3-subsets

$$\left\{ \sum_{n=1}^{\infty} \frac{1}{F_n^{2s}} \quad (s=1,2,3) \right\} , \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n^{2s}} \quad (s=1,2,3) \right\} , \left\{ \sum_{n=1}^{\infty} \frac{1}{L_n^{2s}} \quad (s=1,2,3) \right\} ,$$

$$\left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_n^{2s}} \quad (s=1,2,3) \right\}$$

is algebraically independent over $\mathbb{Q}$. There are

$$\binom{12}{3} = 220$$

3-subsets of Γ. In this paper it is shown that among them there are exactly 22 such 3-subsets which are algebraically dependent over $\mathbb{Q}$. We compute the corresponding polynomials explicitly. For instance, we have

$$2\sum_{n=1}^{\infty} \frac{1}{F_n^2} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n^2} - 5\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_n^2} = 0 \,,$$

$$\left(\sum_{n=1}^{\infty} \frac{1}{F_n^2} \right) \cdot \left(8\sum_{n=1}^{\infty} \frac{1}{L_n^2} + 1 \right) - 5\sum_{n=1}^{\infty} \frac{1}{L_n^2} - 20\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_n^4} = 0 \,.$$

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