ALGEBRAIC RELATIONS FOR RECIPROCAL SUMS OF ODD TERMS IN FIBONACCI NUMBERS

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In this paper, we prove the algebraic independence of the reciprocal sums of odd terms in Fibonacci numbers

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}}\ , \qquad \sum_{n=1}^{\infty}\frac{1}{F_{2n-1}^2}\ , \qquad \sum_{n=1}^{\infty}\frac{1}{F_{2n-1}^3}\ ,$$

and write each

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}^s} \qquad (s\geq 4)$$

as an explicit rational function of these three numbers over $\mathbb{Q}$. Similar results are obtained for various series including the reciprocal sums of odd terms in Lucas numbers.

Key words: Algebraic independence, Fibonacci numbers, Lucas numbers, Jacobian elliptic functions, Ramanujan functions, q-series, Nesterenko's theorem.