ON A SEQUENCE TRANSFORMATION WITH INTEGRAL COEFFICIENTS FOR EULER'S CONSTANT

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Let $s_n = 1+1/2+\ldots+1/(n-1)-\log n$. In 1995, the author has found a series transformation of the type $\sum_{k=0}^{n} \mu_{n,k,\tau} s_{k+\tau}$ with integer coefficients $\mu_{n,k,\tau}$, from which geometric convergence to Euler's constant γ for $\tau = O(n)$ results. In recently published papers *T.Rivoal* and *Kh.&T.Hessami Pilehrood* have generalized this result. In this paper we introduce a series transformation $\sum_{k=0}^{n} \mu_{n,k,\tau_1} s_{k+\tau_2}$ with two parameters τ_1 and τ_2 satisfying $\tau_1 + 1 \leq \tau_2 \leq n + \tau_1 + 1$, and integer coefficients μ_{n,k,τ_1}. By applying the Mellin-Barnes integral representation of the ${}_3F_2$ - function, combinatorial identities, and the analysis of the ψ-function, for $n = 2m$, $\tau_1 = m - 1$ and $\tau_2 = 2m$ we prove that $S := |\sum_{k=0}^{n} \mu_{n,k,\tau_1} s_{k+\tau_2} - \gamma| \leq m/2 \cdot |\zeta(2) - q_m|$, where q_m are explicitly given rational numbers. Finally, $\zeta(2) - q_m$ can be expressed in terms of Legendre-type integrals, which give upper bounds for S. In particular, for $n = 2m$, $\tau_1 = m - 1$ and $\tau_2 = 2m$ this bound equals to $2m \cdot 64^{-m}$.