ÜBER GEWISSE LÖSUNGEN UNIVERSELLER DIFFERENTIALGLEICHUNGEN IN ALGEBRAISCHEN PUNKTEN

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Elementare und Analytische Zahlentheorie, Proceedings ELAZ-Conference, May 24-28, 2004, Schriften der Wissenschaftlichen Gesellschaft 20, Franz Steiner Verlag (2006), 44-56.

It is shown that an effectively computable algebraic differential equation of order five exists such that on the one hand every continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ can be approximated uniformly on $\mathbb{R}$ by a sequence $y_n(x)$ of $C^\infty(\mathbb{R})$-solutions. On the other hand let $1 \leq k_1 < k_2 < \ldots < k_r = K$ be integers. Then, for every index n and for numbers $y_n^{(k_1)}(\tau), \ldots, y_n^{(k_r)}(\tau)$, a linear transcendence measure exists which is effectively computable, i.e. a lower bound exists for

$$\sum_{\rho=1}^{r} \psi_\rho y_n^{(k_\rho)}(\tau)$$

with real algebraic numbers $\psi_1, \ldots, \psi_r$. This measure depends on degrees and heights of τ and $\psi_1, \ldots, \psi_r$, but also on the modulus of continuity of the function f within an interval containing τ and on the distance between $y_n(x)$ and $f(x)$ on $\mathbb{R}$. The proof requires extensive estimates of degrees and heights of polynomials and algebraic numbers which are connected with the explicit solutions $y_n(x)$ of the universal differential equation. The linear transcendence measure is finally given by a quantitative version of Baker's theorem on linear forms in logarithms.