# ON A SEQUENCE TRANSFORMATION WITH INTEGRAL COEFFICIENTS FOR EULER'S CONSTANT, II

Carsten Elsner

**Journal of Number Theory, 124 (2007), 442 - 453**

Let

$$s_1 := 0 , \qquad s_n = 1 + 1/2 + \ldots + 1/(n-1) - \log n \qquad (n \geq 2) .$$

In 1995, the author [1] has found a series transformation of the type $\sum_{k=0}^{n} \mu_{n,k,\tau} s_{k+\tau}$ with integer coefficients $\mu_{n,k,\tau}$, from which geometric convergence to Euler's constant $\gamma$ for $\tau = \mathcal{O}(n)$ results. In recently published papers *T.Rivoal* and *Kh. and T.Hessami Pilehrood* have generalized this result. In the present paper we introduce a series transformation $\sum_{k=0}^{n} \mu_{n,k,\tau_1} s_{k+\tau_2}$ with two parameters $\tau_1$ and $\tau_2$ satisfying $\tau_1 + 1 \leq \tau_2 \leq n + \tau_1 + 1$, and integer coefficients $\mu_{n,k,\tau_1}$. By applying the analysis of the $\psi$ - function, we prove a sharp bound for $|S - \gamma|$. A similar result holds for *generalized Stieltjes constants*. Let $a \geq 0$ be a real number, and let

$$s_1(a) := 0 , \qquad s_n(a) := \Big( \frac{1}{a+1} + \frac{1}{a+2} + \frac{1}{a+3} + \ldots + \frac{1}{a+n-1} \Big) - \log(a+n) \qquad (n \geq 2) .$$

The sequence $\left(s_n(a)\right)_{n\geq 1}$ is convergent for any real number $a > 0$:

$$\lim_{n\to\infty} s_n(a) = \gamma_0(a) - \frac{1}{a} ,$$

where $\gamma_0(a)$ are known as *generalized Stieltjes constants of order 0*, i.e.

$$\gamma_0(a) := -\frac{\Gamma'(a)}{\Gamma(a)} = -\Psi(a) .$$

Particularly, we have for $a = 0$:

$$\lim_{n\to\infty} s_n(0) = \lim_{a\to 0} \Big( \gamma_0(a) - \frac{1}{a} \Big) = \gamma .$$

Our main results are given by the following theorems:

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[1] *On a sequence transformation with integral coefficients for Euler's constant*, Proceedings of the AMS **123** no.5 (1995), 1537-1541

**Theorem 1** *Let $n \geq 1$, $\tau_1 \geq 1$ and $\tau_2 \geq 1$ be integers. Additionally we assume that*

$$1 + \tau_1 \;\leq\; \tau_2 \;.$$

*Then one has*

$$\Big| \sum_{k=0}^{n} (-1)^{n+k} \binom{n+\tau_1+k}{n} \binom{n}{k} \cdot s_{k+\tau_2}(a) - \Big( \gamma_0(a) - \frac{1}{a} \Big) \Big|$$

$$= \int_0^1 \left( \frac{1}{1-u} + \frac{1}{\log u} \right) \cdot u^{a+\tau_2-\tau_1-1} \cdot \frac{d^n}{du^n} \left( \frac{u^{n+\tau_1}(1-u)^n}{n!} \right) du \;.$$

**Theorem 2** *Let $n \geq 1$, $\tau_1 \geq 1$ and $\tau_2 \geq 1$ be integers. Additionally we assume that*

$$1 + \tau_1 \;\leq\; \tau_2 \;\leq\; 1 + n + \tau_1 \;.$$

*Then one has*

$$\Big| \sum_{k=0}^{n} (-1)^{n+k} \binom{n+\tau_1+k}{n} \binom{n}{k} \cdot s_{k+\tau_2}(0) - \gamma \Big|$$

$$= \int_0^1 \int_0^1 w(t) \cdot \frac{(1-u)^{n+\tau_1} u^n (1-t)^{\tau_2-\tau_1-1} t^{n+\tau_1-\tau_2+1}}{(1-ut)^{n+1}} \, du \, dt \;,$$

*with*

$$w(t) \;:=\; \frac{1}{t \cdot \Big( \pi^2 + \log^2(\frac{1}{t} - 1) \Big)} \;.$$

Setting

$$n \;=\; \tau_2 \;=\; dm \;, \qquad \tau_1 = (d-1)m - 1 \qquad (d \geq 2) \;,$$

we get an explicit upper bound from Theorem 2:

**Corollary 1** *For integers $m \geq 2$, $d \geq 3$, we have*

$$\Big| \sum_{k=0}^{dm} (-1)^{dm+k} \binom{(2d-1)m+k-1}{dm} \binom{dm}{k} \cdot s_{k+dm}(0) - \gamma \Big| \;<\; C_d \cdot \left( \frac{\Big(1 - \frac{1}{d}\Big)^d}{(d-1)4^d} \right)^{m-2} ,$$

*where $0 < C_d \leq 1/16\pi^2$ is some constant depending only on $d$. For $d = 2$ one gets*

$$\Big| \sum_{k=0}^{2m} (-1)^{k} \binom{3m+k-1}{2m} \binom{2m}{k} \cdot s_{k+2m}(0) - \gamma \Big| \;<\; \Big( \frac{16}{7\pi} \Big)^2 \cdot \frac{1}{64^m} \qquad (m \geq 1) \;.$$

Our method can be modified to work in general for series transformations connected with Ser-type formulas like

$$\gamma - s_n(0) \;=\; \sum_{m=1}^{\infty} \frac{\Gamma(m)}{(n)_m}\, t_{m+1} \qquad \text{and} \qquad \gamma_0(a) - s_n(a) - \frac{1}{a} \;=\; \sum_{m=1}^{\infty} \frac{\Gamma(m)}{(a+n)_m}\, t_{m+1} \quad (a > 0)$$

with rational numbers $t_{m+1}$ defined by

$$t_{m+1} \;:=\; -\frac{1}{m!} \cdot \int_0^1 (-x)(1-x)(2-x)\cdot \ldots \cdot (m-1-x)\, dx \qquad (m \geq 1)\ .$$

Then, in the case of $\gamma$, we apply the Mellin - Barnes integral representation of the ${}_3F_2$ - function, and use different combinatorial identities. Following this different approach, which is more technically than the method used below, we get somewhat weaker results than that ones stated in Theorem 2 and Corollary 1:

Let $n \geq 1$, $\tau_1 \geq 1$ and $\tau_2 \geq 1$ be integers. Additionally we assume that

$$\tau_1 + 1 \;\leq\; \tau_2 \;\leq\; n + \tau_1 + 1\ .$$

Then one has

$$\Big| \sum_{k=0}^{n} (-1)^{n+k} \binom{n+\tau_1+k}{n} \binom{n}{k} \cdot s_{k+\tau_2} - \gamma \Big|$$

$$\leq\; \frac{(\tau_2-1)!(n+\tau_1)!(n+\tau_1-\tau_2+1)!}{2(n+\tau_1-\tau_2+2)(2n+\tau_1+1)!\tau_1!} \cdot \max_{0\leq x\leq 1} \Big| \sum_{l=0}^{\tau_2-\tau_1-1} \frac{(\tau_1-\tau_2+1)_l (n+\tau_1+1)_l}{l!(\tau_1+1)_l}$$

$$\times\ {}_3F_2 \left( \begin{matrix} n+\tau_1-\tau_2+2 & n+\tau_1-\tau_2+2-x & n-l+1 \\ 2n+\tau_1+2 & n+\tau_1-\tau_2+3 & \end{matrix} \;\Big|\; 1 \right) \Big|$$

For $n = \tau_2 = 2m$, $\tau_1 = m-1$ we then have for every integer $m \geq 1$ that

$$\Big| \sum_{k=0}^{2m} (-1)^{k} \binom{3m+k-1}{2m} \binom{2m}{k} s_{k+2m} - \gamma \Big|$$

$$\leq\; \frac{m}{2} \cdot |\zeta(2) - q_m| \;=\; \frac{m}{2} \cdot \int_0^1 \int_0^1 \frac{z^{m-1} w^{2m} (1-z)^{m+1} (1-w)^{3m-1}}{(1-wz)^{2m+1}}\, dw\, dz \;\leq\; \frac{2m}{64^m}\ ,$$

where

$$q_m \;:=\; \sum_{k=0}^{2m} (-1)^{k} \binom{3m+k-1}{2m} \binom{2m}{k} \cdot \sum_{\nu=1}^{k+2m} \frac{1}{\nu^2}\ .$$

Here a Beuker's type integral for $\zeta(2)$ pops up.