APPROXIMATION OF VALUES OF HYPERGEOMETRIC FUNCTIONS BY RESTRICTED RATIONALS

Carsten Elsner
FHDW Hannover, University of Applied Sciences,
Freundallee 15, D-30173 Hannover, Germany

Takao Komatsu
Department of Mathematical System Science,
Faculty of Science and Technology, Hirosaki University,
Hirosaki 036-8561, Japan

Iekata Shiokawa
Department of Mathematics, Keio University,
3-14-1 Hiyoshi, Kohuku-ku, Yokohama 223-8522, Japan

Journal de Théorie des Nombres de Bordeaux 19 (2007), 393 - 404.

We compute upper and lower bounds for the approximation of hyperbolic functions at points $1/s$ $(s = 1, 2, \ldots)$ by rationals x/y, such that x, y satisfy a quadratic equation. For instance, all positive integers x, y with $y \geq 3$ solving the Pythagorean equation $x^2 + y^2 = z^2$ satisfy

$$|y \sinh(1/s) - x| \gg \frac{\log\log y}{\log y} .$$

Conversely, for every $s = 1, 2, \ldots$ there are infinitely many coprime integers x, y, such that

$$|y \sinh(1/s) - x| \ll \frac{\log\log y}{\log y}$$

and $x^2 + y^2 = z^2$ hold simultaneously for some integer z. A generalization to the approximation of $h(e^{1/s})$ for rational functions $h(t)$ is included.