ON THE APPROXIMATION OF REAL CONTINUOUS FUNCTIONS BY SERIES OF SOLUTIONS OF A SINGLE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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Let $\omega \in \mathbb{R}^s \to \mathbb{R}_{>0}$ denote some bounded continuous weight function taking positive values everywhere, such that additionally

$$\int_{\mathbb{R}^s} \omega(x_1,\ldots,x_s)\,d\mathbf{x} \,=\, 1$$

holds. For the sake of brevity we use the notation $d\mathbf{x}$ to express $dx_1 dx_2 \ldots dx_s$. Now let

$$\|g\|_\omega \,:=\, \int_{\mathbb{R}^s} \omega(x_1,\ldots,x_s)|g(x_1,\ldots,x_s)|\,d\mathbf{x} \qquad (g \in C(\mathbb{R}^s))\ .$$

Theorem 1 *There exists an effectively computable polynomial $P(x,t_0,\ldots,t_5) \in \mathbb{Z}[x,t_0,\ldots,t_5]$ having the following property.*
Let $s \geq 1$ be some integer, let $f \in \mathbb{R}^s \to \mathbb{R}$ be some continuous function defined on $\mathbb{R}^s$ and let ε be some arbitrary positive number. Then a series $H \in C^\infty(\mathbb{R}^s)$ of analytic functions $H_r \in C^\omega(\mathbb{R}^s)$ exists, say

$$H(x_1,\ldots,x_s) \,=\, \sum_{r=1}^{\infty} H_r(x_1,\ldots,x_s) \qquad (x_\nu \in \mathbb{R}\ ;\ \nu = 1,\ldots,s)\ ,$$

such that $\|f-H\|_\omega < \varepsilon$ holds, and every analytic function H_r solves the system of partial differential equations

$$P\Big(x_\sigma\,;\,H_r\,,\,\frac{\delta H_r}{\delta x_\sigma}\,,\,\ldots\,,\,\frac{\delta^5 H_r}{\delta x_\sigma^5}\Big) \,=\, 0 \qquad (\sigma = 1,\ldots,s)\ . \tag{1}$$

A specific polynomial $P(x,t_0,\ldots,t_5)$ is homogeneous of degree 16 in its variables $t_0,\ldots,t_5$, and it consists of 575 terms of the form

$$a \cdot x^b \cdot t_0^{c_0} \cdot \ldots \cdot t_5^{c_5} \qquad (a,b,c_0,\ldots,c_5 \in \mathbb{Z}\ ;\ b,c_0,\ldots,c_5 \geq 0\,;\ c_0+\ldots+c_5 = 16)\ .$$

Using standard arguments it follows easily from (1) that H_r also satisfies a system of autonomous partial differential equations of order six.