ON A METHOD TO TRANSFORM ALGEBRAIC DIFFERENTIAL EQUATIONS INTO UNIVERSAL EQUATIONS

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Definition 1 *Let C be a class of real-valued functions defined on $D \subseteq \mathbb{R}$. An ADE is called universal with respect to $C(D)$, if every continuous function on D can be approximated uniformly by C-solutions of this ADE on D.*

The transformation algorithm requires the definition of certain rational numbers given by the following identity. Let $h : C^\infty(\mathbb{R}) \to \mathbb{R}_{>0}$ be a real function, and let $k \geq 0$ and $n \geq 1$ be some integers. Then the k-th derivative of the function $g(x) := (h(x))^{\frac{1}{n}}$ takes the form

$$g^{(k)}(x) = \sum_{0 \leq a_1, a_2, \ldots, a_k \leq k} p(n; a_1, \ldots, a_k) \cdot (h(x))^{\frac{1}{n} - (a_1 + \ldots + a_k)} \cdot (h'(x))^{a_1} \cdot \ldots \cdot (h^{(k)}(x))^{a_k}$$

with specific rationals $p(n; a_1, \ldots, a_k)$, which are uniquely determined by the above identity. Particularly one computes the rationals $p \neq 0$ for $k = 1, 2, 3, 4$ and $n \geq 1$:
k=1:

$$p(n; 1) = \frac{1}{n} \;;$$

k=2:

$$p(n; 0, 1) = \frac{1}{n}\;, \quad p(n; 2, 0) = \frac{1-n}{n^2}\;;$$

k=3:

$$p(n; 0, 0, 1) = \frac{1}{n}\;, \quad p(n; 1, 1, 0) = \frac{3-3n}{n^2}\;, \quad p(n; 3, 0, 0) = \frac{2n^2 - 3n + 1}{n^3}\;;$$

k=4:

$$p(n; 0, 0, 0, 1) = \frac{1}{n}\;, \quad p(n; 0, 2, 0, 0) = \frac{3-3n}{n^2}\;, \quad p(n; 1, 0, 1, 0) = \frac{4-4n}{n^2}\;,$$

$$p(n; 2, 1, 0, 0) = \frac{12n^2 - 18n + 6}{n^3}\;, \quad p(n; 4, 0, 0, 0) = \frac{1 - 6n + 11n^2 - 6n^3}{n^4}\;.$$

We now state the main results.

Theorem 1 *Let $n > m \geq 1$ and $k \geq 0$ be integers. Moreover, let $P(y, y', y'', \ldots, y^{(m)}) = 0$ be an autonomous homogeneous ADE of weight w such that there is an interval $[\alpha; \beta]$ and a solution $u = y \in C^n((\alpha; \beta))$ of the ADE satisfying*

(i) $u(x) > 0 \qquad (\alpha < x < \beta)$ *;*

(ii) $u(x) = O(|x - \alpha|^{r_1})$ *(for $x \to \alpha$) ,* $u(x) = O(|x - \beta|^{r_2})$ *(for $x \to \beta$)* *with certain integers $r_1, r_2 \geq 1$.*

(iii) *One substitutes 1 for y and*

$$\sum_{1 \leq \kappa \leq k} \sum_{0 \leq a_1, \ldots, a_k \leq k} p(n; a_1, \ldots, a_k) \cdot (f(x))^{k-\kappa} \cdot (f'(x))^{a_1} \cdot \ldots \cdot (f^{(k)}(x))^{a_k}$$

for $y^{(k)}$ $(1 \leq k \leq m)$ into $P(y, y', y'', \ldots, y^{(m)}) = 0$.

(iv) *After that put $f(x) = z', f'(x) = z'', \ldots, f^{(m)}(x) = z^{(m+1)}$.*

The resulting ADE in $z', z'', \ldots, z^{(m+1)}$ is a universal ADE with respect to $C^n(\mathbb{R})$. It is of weight $2w$ and homogeneous of degree w.

Examples:

(1.) The simple homogeneous ADE $y''' = 0$ of weight 3 is solvable by $u = 1 - x^2$. The transformation algorithm from Theorem 1 yields

$$\frac{1}{n} f^2 f''' + \frac{3 - 3n}{n^2} f f' f'' + \frac{2n^2 - 3n + 1}{n^3} f'^3 \;=\; 0 \;.$$

Multiplying by n^3 and substituting z' for f, one gets the ADE

$$n^2 z'^2 z'''' - 3n(n-1) z' z'' z''' + (2n^2 - 3n + 1) z''^3 = 0 \;.$$

This is a universal ADE already known to DUFFIN (*Proc.Nat.Acad.Sci. U.S.A.* **78** (1981), no.8, part 1, 4661-4662.)

(2.) The application of the transformation method to the equation $yy''' - 2y'y'' = 0$ of weight 3 gives the following result.

Theorem 2 *The differential equation*

$$n^2 y'^2 y^{(4)} - n(3n-1) y' y'' y''' + (n-1)(2n+1) y''^3 \;=\; 0$$

is universal with respect to $C^n(\mathbb{R})$ for $n > 3$.

(3.) Finally a result concerning algebraic properties of the solutions of a universal ADE is given.

Theorem 3 *For any integer $n > 3$ the differential equation*

$$n y'^2 y^{(4)} - (3n-2) y' y'' y''' + 2(n-1) y''^3 \;=\; 0$$

is universal with respect to $C^n(\mathbb{R})$. Furthermore, every continuous function defined on the real line can be approximated uniformly by a sequence $(y_r)_{r \geq 1}$ of $C^n(\mathbb{R})$-solutions of the ADE in such a way that every function $y = y_r$ has the following properties.

Let $0 \leq k_1 < k_2 < \ldots < k_s \leq n$ *be integers, and let* $q_1, \ldots, q_s$ *be rationals such that none of the numbers*

$$y^{(k_1)}(q_1)\,,\; y^{(k_2)}(q_2)\,, \ldots,\; y^{(k_s)}(q_s) \tag{1}$$

vanishes. Then the family of numbers

$$1\,,\; y^{(k_1)}(q_1)\,,\; y^{(k_2)}(q_2)\,, \ldots,\; y^{(k_s)}(q_s)$$

is linearly independent over the field of real algebraic numbers, provided that in the case of an even integer n *the equations* $k_1 = 0$ *and* $k_2 = 1$ *do not occur simultaneously. Particularly all the numbers from (1) are transcendental.*