ON THE APPROXIMATION OF CONTINUOUS FUNCTIONS BY ANALYTIC SOLUTIONS OF UNIVERSAL FUNCTIONAL-DIFFERENTIAL EQUATIONS

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We first consider continuous functions f on $\mathbb{R}$ where both the limits

$$\lim_{x\to-\infty} f(x) \quad \text{and} \quad \lim_{x\to+\infty} f(x) \tag{1}$$

exist in $\mathbb{R}$. One gets:

Theorem 1 *There exists a nontrivial sixth-order ADE of the form*

$$P(x; y', y'', \dots, y^{(6)}) \,=\, 0 \tag{2}$$

such that any real continuous function f on the real line can be uniformly approximated by the real analytic solutions of (2), provided that the limits in (1) exist in $\mathbb{R}$.

Corollary 1 *There exists a nontrivial ADE of the form*

$$P(y', y'', \dots, y^{(7)}) \,=\, 0$$

such that the real-analytic solutions have the same property as the real-analytic solutions of the ADE (2).

Theorem 2 *There exists a nontrivial AFE of the form*

$$P(\, y'(x), y'(x+\log 2), \dots, y'(x+5\log 2)\,) \,=\, 0 \tag{3}$$

such that for any compact interval I the real-analytic solutions of (3) (defined on the whole real line) are dense in $C(I)$.

Theorem 3 *Let n denote some positive integer, and let P be some non-zero polynomial having real coefficients. Then, for every positive real number Δ and any compact interval $I \subset \mathbb{R}$ satisfying $|I| > 2n\Delta$, the continuous solutions g of the AFE*

$$P(\, g(x), g(x+\Delta), \dots, g(x+n\Delta)\,) \,=\, 0 \qquad (x \pm n\Delta \in I)$$

are not dense in $C(I)$.

By $M(\mathbb{C})$ we denote the set of meromorphic functions f defined on $\mathbb{C}$ such that additionally two conditions are satisfied:
(i) $f(z)$ *is analytically at every point* z *with* $\Im z = 0\,,\ \pm 2\pi\,,\ \pm 4\pi$.
(ii) $f(z)$ *takes real values at points* z *from the real axis.*
Particularly every function $f(z)$ from $M(\mathbb{C})$, where z is restricted on the real line, represents a real-valued analytical function from $C^{\omega}(\mathbb{R})$, and there is some real-valued analytical antiderivative defined on the real axis.

Theorem 4 *Let* I *denote a compact interval. Then every function from* $C(I)$ *can be uniformly approximated by real-valued analytic functions* y *defined on* I*, such that* $y' \in M(\mathbb{C})$ *holds and* y' *satisfies the AFE*

$$y_1y_2^2y_3y_5 - y_1y_2^2y_4^2 - y_1y_2^2y_4y_5 + y_1y_2y_3y_4^2 + y_1y_2y_4^2y_5$$
$$-y_1y_3y_4^2y_5 - y_2^2y_3y_4y_5 + y_2^2y_4^2y_5 \;=\; 0 \qquad (4)$$

with

$$y_1 := y'(x-4\pi i)\,,\ y_2 := y'(x-2\pi i)\,,\ \ ,y_3 := y'(x)\,,$$
$$y_4 := y'(x+2\pi i)\,,\ y_5 := y'(x+4\pi i)\ .$$

The identity in (4) represents an AFE of order one for real-valued analytical functions on arbitrary compact intervals I.