ON RATIONAL APPROXIMATIONS BY PYTHAGOREAN NUMBERS

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Theorem 1 *Let $\xi > 0$ denote a real irrational number such that the quotients of the continued fraction expansion of at least one of the numbers $\eta_1 := \xi + \sqrt{1+\xi^2}$ and $\eta_2 := (1+\sqrt{1+\xi^2})/\xi$ are not bounded. Then there are infinitely many triplets of positive integers x, y, z satisfying*

$$|\xi y - x| \,=\, o(1) \quad \textit{and} \quad x^2 + y^2 = z^2 \ .$$

Conversely, if the quotients of both of the numbers η_1 and η_2 are bounded, then there exists some $\delta > 0$ such that

$$|\xi y - x| \,\geq\, \delta$$

holds for all positive integers x, y, z where $x^2 + y^2 = z^2$.

Corollary 1 *To almost all real numbers ξ (in the sense of the Lebesgue measure) there are infinitely many triplets of integers $x \neq 0,\ y > 0, z > 0$ satisfying*

$$|\xi y - x| \,=\, o(1) \quad \textit{and} \quad x^2 + y^2 = z^2 \ .$$

Corollary 2 *Let $r > 1$ denote some rational such that $\xi := \sqrt{r^2-1}$ is an irrational number. Then the inequality*

$$|\xi y - x| \,>\, \delta$$

holds for some $\delta > 0$ (depending only on r) and for all positive integers x, y, z with $x^2 + y^2 = z^2$.

Theorem 2 *Let $k \geq 2$ denote an even integer, and $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$ are the Fibonacci numbers. Then,*

$$0 \,<\, \frac{F_k\sqrt{5}}{2}\cdot(2F_nF_{n+k}) - F_kF_{2n+k} + (-1)^n\frac{F_{2k}}{\sqrt{5}} \,<\, \frac{2^{2n+1}}{5(1+\sqrt{5})^{2n}}$$

holds for all integers $n \geq 1$, and we have

$$(2F_nF_{n+k})^2 + (F_kF_{2n+k})^2 \,=\, (F_{n+k}^2 + F_n^2)^2 \quad (n \geq 1)\ .$$